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Hyperbolic billiards of pure $D = 4$ supergravities

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Abstract: We compute the billiards that emerge in the Belinskii-Khalatnikov-Lifshitz (BKL) limit for all pure supergravities in $D = 4$ spacetime dimensions, as well as for $D = 4$, $N = 4$ supergravities coupled to k ($N = 4$) Maxwell supermultiplets. We find that just as for the cases $N = 0$ and $N = 8$ investigated previously, these billiards can be identified with the fundamental Weyl chambers of hyperbolic Kac-Moody algebras. Hence, the dynamics is chaotic in the BKL limit. A new feature arises, however, which is that the relevant Kac-Moody algebra can be the lorentzian extension of a *twisted* affine Kac-Moody algebra, while the $N = 0$ and $N = 8$ cases are untwisted. This occurs for $N = 5$, where one gets $A_4^{(2)\wedge}$, and for $N = 3$ and 2, for which one gets $A_2^{(2)\wedge}$. An understanding of this property is provided by showing that the data relevant for determining the billiards are the *restricted root system* and the *maximal split subalgebra* of the finite-dimensional real symmetry algebra characterizing the toroidal reduction to $D = 3$ spacetime dimensions. To summarise: split symmetry controls chaos.

Keywords: M-Theory, String Duality, Space-Time Symmetries, Supergravity Models.

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1. Introduction

As it has been shown recently, the classical dynamics of the spatial scale factors and of the dilaton(s) (if any) of D-dimensional gravity coupled to p-forms and scalar fields can be described, in the vicinity of a spacelike singularity, as a billiard motion in a region of hyperbolic space bounded by hyperplanes [1].[1] This generalizes known results for pure gravity in $D = 4$ spacetime dimensions [2]–[4]. Furthermore, in the case of the bosonic sector of 11-dimensional supergravity or 10-dimensional supergravities, the relevant billiard turns out to be identifiable with the fundamental Weyl chamber of the Kac-Moody algebras E_{10}, BE_{10} or DE_{10} [1] — which are respectively the overextensions [5, 6] of E_8, B_8 and D_8, i.e., $E_{10} \equiv E_8^{\wedge\wedge}$, $BE_{10} \equiv B_8^{\wedge\wedge}$ and $DE_{10} \equiv D_8^{\wedge\wedge}$ — while for D-dimensional pure gravity, the algebra is the overextension $A_{D-3}^{\wedge\wedge}$ of A_{D-3} [7]. The geometrical reflexions occurring when the system hits the billiard walls are fundamental Weyl reflexions and the motion can thus be identified with an (infinite) Weyl word. The fact that the underlying Kac-Moody algebras are hyperbolic (provided $D < 11$ for pure gravity) explains [7] the chaotic behaviour of these systems as one approaches the singularity [8, 9].

[1] As it has become standard practice in the field, the word *billiard* used as a noun in the singular denotes the dynamical system consisting of a ball moving freely on a "table" (region in some riemannian space), with elastic bounces against the edges. *Billiard* also sometimes means the table itself.

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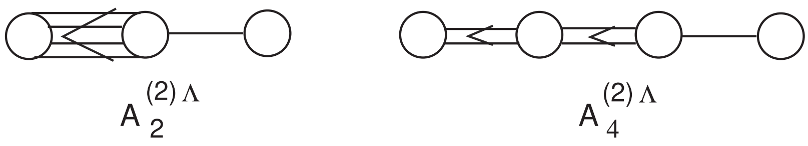

Figure 1: Hyperbolic Kac-Moody algebras $A_2^{(2)\wedge}$ and $A_4^{(2)\wedge}$.

When reduced to four spacetime dimensions, the above models correspond to pure $N = 8$ supergravity, or to $N = 4$ supergravity coupled to a collection of $N = 4$ Maxwell/Yang-Mills multiplets (6 Maxwell multiplets for pure $N = 4$, $D = 10$ supergravity, $6 + 16$ vector multiplets for $N = 4$, $D = 10$ supergravity with $E_8 \times E_8$ or SO(32) Yang-Mills field). Only the $N = 8$ case defines a pure supergravity theory in four spacetime dimensions. The purpose of this article is to investigate systematically the billiards that describe the dynamics of all pure $D = 4$ supergravities ($N = 1, 2, 3, 4, 5, 6, 8$) in the vicinity of a spacelike singularity. We also investigate $D = 4, N = 4$ supergravity coupled to a collection of an arbitrary number k of $N = 4$ vector multiplets.

We find that the billiards for all these models can also be associated with hyperbolic Kac-Moody algebras. Furthermore, we prove that "split symmetry controls chaos" in the following sense: let $\mathcal{U}_3$ be the finite-dimensional real U-duality algebra that appears in the toroidal compactification of the theory to 3 dimensions. Then, the hyperbolic Kac-Moody algebra whose fundamental Weyl chamber determines the billiard is the overextension of the "maximal split subalgebra" $\mathcal{F}$ of $\mathcal{U}_3$ (and not of $\mathcal{U}_3$ itself except when $\mathcal{U}_3$ and $\mathcal{F}$ coincide, which occurs only when $\mathcal{U}_3$ is split, i.e., maximally non-compact). This explains in particular why it is $BE_{10} \equiv B_8^{\wedge\wedge}$ (rather than the overextension of the split form D_{16} of the non-split $so(8, 8+16)$) that determines the heterotic billiard. However, while the Kac-Moody algebras of the previously studied models are given by standard overextensions of finite-dimensional Lie algebras, some of the theories investigated here are characterized by a new feature: the extension involves a "twist". Specifically, the twist occurs for $N = 2, 3$ and 5, for which one gets respectively the lorentzian extensions $A_2^{(2)\wedge}$ and $A_4^{(2)\wedge}$ of the twisted affine algebras $A_2^{(2)}$ and $A_4^{(2)}$ with Dynkin diagrams in figure 1. Note that the role of B_8 versus $so(8, 8 + 16)$ was understood in the context of non-split U-duality for the $so(8, 8 + r)$-models in [10]. In that precise case, there is no twist.

In the next section, we set up our conventions and terminology. Then, we derive the central theorem that relates the Kac-Moody algebra $\mathcal{A}$ whose fundamental Weyl chamber is the billiard table to the maximal split subalgebra $\mathcal{F}$ and the *restricted* root system of the *real* symmetry algebra $\mathcal{U}_3$ appearing in 3 spacetime dimensions (section 3). We show quite generally that $\mathcal{A}$ is the overextension of $\mathcal{F}$, with a twist only when the root system is of *bc*-type. The theorem covers all pure supergravity models in $D = 4$ spacetime dimensions, as well as $N = 4, D = 4$ SUGRA with k Maxwell multiplets. So when the U-duality algebra $\mathcal{U}_3$ is not split (i.e. not maximally non-compact) its maximal split subalgebra $\mathcal{F}$, which controls chaos, is smaller. The analysis also explains the wall multiplicities. In section 4, the billiard is computed in the particular case of $N = 2$ supergravity directly in $D = 4$ dimensions, without going to $D = 3$ dimensions. This sheds a different light on the twist.

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Finally, we relate the occurrence of the twist to a property of real forms of untwisted but non split affine Kac-Moody algebras [11].

2. Conventions

Let $\mathcal{G}_{\mathbb{C}}$ be a complex, finite-dimensional Lie algebra and let $\mathcal{G}$ be one of its real forms. We denote by $\mathcal{G}^{\wedge}$ the corresponding untwisted algebra of currents (where all the currents are integer-moded). It is a real form of the untwisted affine extension of $\mathcal{G}_{\mathbb{C}}$, defined from the Chevalley-Serre presentation by adding generators associated with the new, "affine" root $\alpha_0 = \delta - \theta$, where θ is the highest root of $\mathcal{G}_{\mathbb{C}}$ and δ the null root of $(\mathcal{G}_{\mathbb{C}})^{\wedge}$ (see [12, 13]; $(\mathcal{G}_{\mathbb{C}})^{\wedge}$ is $\mathcal{G}_{\mathbb{C}}^{(1)}$ in Kac's notations). We shall also consider real twisted affine algebras, but only in their split form. These are again defined from the standard Dynkin diagrams in terms of Chevalley-Serre generators and relations, but one considers only real combinations. We adopt Kac's notations [12] in the twisted case. Given a split (untwisted or twisted) affine Kac-Moody algebra $\mathcal{E}$, we define its real lorentzian extension $\mathcal{E}^{\wedge}$ by adding an "overextended root" [5, 6] to its Dynkin diagram and considering the corresponding set of generators and relations over the reals. The overextended root is attached to the affine root α_0 with a single line in the untwisted case. For the twisted algebras $A_{2\ell}^{(2)}$ — the only twisted cases we shall encounter here — the overextended root is attached to the unique root carrying label 1, which is the longest root. Note that this longest root is *not* the root α_0 of [12]. More information on overextensions is given in the appendix A.

To characterize the real form $\mathcal{G}$, we adopt the Tits-Sakate theory as developed for instance in [14]. One selects a maximally non-compact Cartan subalgebra of $\mathcal{G}$ and one diagonalizes simultaneously in $\mathcal{G}$ all the operators ad_h with h in this Cartan subalgebra. This defines the restricted roots and the restricted root system $\bar{B}$, which can be either one of the standard reduced root systems a_n, b_n, c_n, d_n, g_2, f_4, e_6, e_7 or e_8, or one of the nonreduced bc_n systems. The bc_n root system is obtained by merging the b_n and c_n root systems in such a way that the long roots of b_n are the short roots of c_n. It is called nonreduced because α and 2α can be simultaneously roots (if α is a short root of b_n, 2α is a long root of c_n). The restricted roots include all the standard roots of a Lie algebra, which we call the "maximal split subalgebra" following Borel and Tits [15] and denote by $\mathcal{F}$. Note, however, that there are in general crucial differences between the restricted root system of $\mathcal{G}$ and the standard root system of $\mathcal{F}$: (i) twice a root of $\mathcal{F}$ can be a restricted root and (ii) the restricted roots can come with a non trivial multiplicity. The dimension of the maximally non-compact Cartan subalgebra of $\mathcal{G}$, which is the Cartan subalgebra of $\mathcal{F}$, is called the real rank of $\mathcal{G}$ and denoted by r. It is only if $\mathcal{G}$ is the split real form of $\mathcal{G}_{\mathbb{C}}$ that $\mathcal{F}$ coincides with $\mathcal{G}$. This information is encoded in the Tits-Sakate diagrams and given for instance in [16, table VI, chapter X]. The importance of non split real forms of U-duality groups dates back to supergravity days [5]. The facts that its precise real form is related to the $SL(D-d, R)$ compactification symmetry and that this fixes the maximal oxidation possible have also been used and exposed repeatedly.

3. A general theorem

When reduced to $D = 3$ spacetime dimensions, the bosonic sectors of all $D = 4$ pure supergravity theories become the Einstein theory coupled to a coset model G/H where H is the maximal compact subgroup of G, and where G depends on N [17]. We give in the first column of table I of the concluding section the (real) Lie algebra $\mathcal{G} \equiv \mathcal{U}_3$ of the $D = 3$ symmetry group G. The information is taken from [17].

Now, the computation of the billiard can be carried out in any number of dimensions ≥ 3 because the dominant walls that define the billiard are invariant under toroidal dimensional reduction and dualization [18]. If the theory is explicitly known in 3 dimensions, which is the case here, the easiest way to uncover the underlying Kac-Moody structure is to analyze the billiard in that dimension, where it is particularly transparent. In this section, we determine the billiard for general coset models coupled to gravity in 3 dimensions, where $\mathcal{G}$ is not necessarily the split real form of $\mathcal{G}_{\mathbb{C}}$. Our theorem will be a generalization to the non-split case of the results obtained in [18] for the split case, where the lagrangians of [19, 20] of maximally non-compact coset models were investigated.

The basic tool will be the Iwasawa decomposition of the non compact group G where the restricted root system plays a central rôle (see e.g. [16], chapter IX). Using this decomposition, the G/H coset action takes the form

$$S^{\phi^\alpha,\chi^A} = -\int g^{\mu\nu}\sqrt{-g}\Big(\sum_\alpha \partial_\mu\phi^\alpha\partial_\nu\phi^\alpha + \frac{1}{2}\sum_A e^{2\lambda^A(\phi)}(\partial_\mu\chi^A+\cdots)(\partial_\nu\chi^A+\cdots)\Big)d^3x\,, \quad (3.1)$$

where the ellipsis denote "correction terms" to the "abelian curvatures" $d\chi^A$. In (3.1), the ϕ^α's are the dilatons, whose number is equal to the real rank r of $\mathcal{G}$. The dilatons can be identified with coordinates on the maximally non-compact Cartan subalgebra of $\mathcal{G}$. The linear forms $\lambda^A(\phi)$ are the restricted roots (taking into account multiplicities, i.e., if a restricted root has multiplicity k, there are then k linear forms $\lambda^A(\phi)$ associated with it, say $\lambda^1(\phi), \cdots, \lambda^k(\phi)$, which are equal, $\lambda^1(\phi) = \lambda^2(\phi) = \cdots = \lambda^k(\phi)$). There is an axion field χ^A for each linear form $\lambda^A(\phi)$. We shall denote by $\theta(\phi)$ the highest root of the restricted root system, hereafter called $\bar{B}$.

The complete action for the system, including gravity, is the sum of (3.1) and of the Einstein action

$$S^E = \int \sqrt{-g}\, R\, d^3x, \quad (3.2)$$

i.e.,

$$S = S^E + S^{\phi^\alpha,\chi^A}\,. \quad (3.3)$$

As in [9], we normalize the dilaton kinetic term such that it has weight one with respect to the Einstein term. To get the billiard walls, one decomposes the 2-dimensional spatial metric g_{ij} as[2]

$$g_{ij} = \begin{pmatrix} e^{-2\beta^1} & ne^{-2\beta^1} \\ ne^{-2\beta^1} & n^2e^{-2\beta^1} + e^{-2\beta^2} \end{pmatrix}. \quad (3.4)$$

[2]The rules for writing down the billiards have been stated in [9, 1, 7]. A systematic derivation is presented in [21].

We shall call collectively "(logarithmic) scale factors" both the β^i's ($i = 1, 2$) and the dilatons ϕ^α. The action determines a metric in the space of the scale factors which reads, in our normalization

$$d\sigma^2 = \sum_{i=1}^{2} (d\beta^i)^2 - \left(\sum_{i=1}^{2} d\beta^i \right)^2 + \sum_{\alpha=1}^{r} (d\phi^\alpha)^2 \tag{3.5}$$

The inverse metric is

$$(\partial f | \partial f) = \sum_{i=1}^{2} (\partial_i f)^2 - \left(\sum_{i=1}^{2} \partial_i f \right)^2 + \left(\sum_{\alpha=1}^{r} \partial_\alpha f \right)^2 . \tag{3.6}$$

The normalization of the roots of the restricted root system is such that the highest root θ has length squared equal to 2,

$$(\theta|\theta) = 2 . \tag{3.7}$$

The reason for this will be given below.

The linear wall forms defining the billiard associated with the action (3.3) are the following [1, 21]

- Axion electric walls, coming from the kinetic energy $(\dot{\chi}^A)^2$ of the axions

$$w_E^A = \lambda^A(\phi) . \tag{3.8}$$

- Axion magnetic walls, coming from the potential energy $(\partial_k \chi^A)^2$ of the axions

$$w_M^{A,i} = \beta^i - \lambda^A(\phi) . \tag{3.9}$$

- Symmetry wall, coming from the kinetic term of the Iwasawa parameter n in (3.4)

$$w_S = \beta^2 - \beta^1 . \tag{3.10}$$

The same electric or magnetic wall forms may occur several times, but this does not affect the analysis. The billiard is defined by the inequalities $w_\Gamma \geq 0$ where Γ runs over all wall forms (if one inequality occurs several times, we clearly only need to keep it once, which is why multiplicities of wall forms are not important here). The walls are $w_\Gamma = 0$. Since some of the wall forms can be expressed as linear combinations with non-negative (integer) coefficients of a smaller subset of wall forms, only this smaller subset is relevant. The relevant subset is easily determined to contain

1. The electric wall form

$$\sigma_\alpha(\phi) , \tag{3.11}$$

   where the σ_α are the simple roots of the restricted root system $\bar{B}$.

2. The magnetic wall form

$$\beta^1 - \theta(\phi) , \tag{3.12}$$

   where $\theta(\phi)$ is the highest root of $\bar{B}$.

3. The symmetry wall (3.10).

We denote collectively the dominant wall forms by α_i $(i = 1, \ldots, r, r+1, r+2)$.

The dominant wall forms are identified with the simple roots of the searched-for Kac-Moody algebra [1, 7]. Once the dominant wall forms have been determined, one simply computes the Cartan matrix of the Kac-Moody algebra through the familiar formula

$$A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} . \tag{3.13}$$

The following theorem is a straightforward consequence of the previous considerations and constitutes the core result of our paper.

Theorem. *(i) If the restricted root system is not of bc-type, the billiard is the fundamental Weyl chamber of the Kac-Moody algebra $\mathcal{F}^{\wedge\wedge}$ (overextension of the maximal split subalgebra $\mathcal{F}$);*

(ii) If the restricted root system is bc_r, the billiard is the fundamental Weyl chamber of the Kac-Moody algebra $A_{2r}^{(2)\wedge}$.

Proof. One must compute the Cartan matrix.

- In the first step, we determine the "electric" submatrix of the Cartan matrix obtained by restricting (3.13) to the dominant electric wall forms (3.11). This clearly yields the Cartan matrix of the maximal split subalgebra $\mathcal{F}$ since the σ_α are the simple roots (of both $\bar{B}$ and $\mathcal{F}$).

- We next add the dominant magnetic wall form (3.12).

  – If the restricted root system is not of bc-type, the highest root $\theta(\phi)$ is also the highest root of the root system of $\mathcal{F}$. Since the linear form β^1 has length squared equal to zero and is orthogonal to any linear form involving only the dilatons, we see that the dominant magnetic wall form (3.12) has length squared equal to two and is such that $(\beta^1 - \theta|\sigma_\alpha) = -(\theta|\sigma_\alpha)$. This enables one to identify the dominant magnetic wall form with the affine root of the untwisted affine extension $\mathcal{F}^\wedge$ of $\mathcal{F}$ [12, 13]. At this stage, we thus have $\mathcal{F}^\wedge$.

  – If the restricted root system is of bc-type, say bc_r, the first step yields the Cartan matrix of b_r. We order the simple roots of $\bar{B}$ so that σ_1 is the short root and σ_i is linked to σ_{i-1} and σ_{i+1} $(1 < i < r)$. The highest root θ of the bc_r-system is connected only to σ_r and its length squared is four times that of a short root ([16], chapter X). This yields the Cartan matrix of the twisted affine algebra $A_{2r}^{(2)}$ (which one might in fact denote, with adapted conventions, $BC_r^\wedge$, as in [13]). Note that since the highest root has squared length equal to 2, the short root σ_1 has squared length 1/2, the other simple roots having squared length equal to 1.

- What remains to be done is to add the symmetry wall. The only non-vanishing scalar products involving w_S are $(w_S|w_S) = 2$ and $(w_S|\beta^1 - \theta) = -1$. Thus w_S is

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attached by a single line to the dominant magnetic root $\beta^1 - \theta$. This yields in all cases the lorentzian extension of the affine algebra obtained in the previous step, the overextended root being w_S. Thus we do indeed get $\mathcal{F}^{\wedge\wedge}$ if the restricted root system is not of bc-type, and $A_{2k}^{(2)\wedge}$ otherwise. $\square$

In the particular case when $\mathcal{G}$ is the split real form of $\mathcal{G}_{\mathbb{C}}$, one has $\mathcal{G} = \mathcal{F}$ and the restricted root system coincides with the root system of $\mathcal{G}$ (which is reduced and for which each root has multiplicity one). The relevant Kac-Moody algebra is then $\mathcal{G}^{\wedge\wedge}$ as a particular case of (i) in the theorem. This result has been established previously in [18] along similar 3-dimensional lines. The reason that $(\theta|\theta) = 2$ is also the same as in that paper. It comes from the fact that the theory is the reduction of a higher-dimensional one. Indeed, the dominant magnetic wall (3.12) is a symmetry wall in higher dimensions[3] and these walls have all length squared equal to two [1]. This normalization of the roots of the coset space might not hold for theories that cannot be oxidized [20] (see also [22] for a group-theory approach to oxidation). If the normalization of θ were changed, one would not get the Kac-Moody algebras listed below.

Since all the supergravity models under consideration fall within the scope of the above theorem, it is now immediate to determine the associated billiard Kac-Moody algebra. The results are collected in table I of the concluding section. Note, as a further check, that the highest root $\theta(\phi)$ is always non degenerate for these models (see [16, table VI in chapter X]). This is necessary because we have seen that $\beta^1 - \theta$ is a symmetry wall in higher dimensions, and symmetry walls are never degenerate [21]. A highest root with multiplicity > 1 is thus an obstruction to oxidation. Note also that all the algebras $\mathcal{A}$ of the table are hyperbolic, so that the models are all chaotic.

4. $N = 2$ pure supergravity

In order to understand better the twist, we shall repeat the billiard computation in $D = 4$ spacetime dimensions for the simplest case where the twist is present, namely, $N = 2$ pure supergravity. The bosonic sector is then the Einstein-Maxwell theory with one Maxwell field. Following the rules of [1, 9], one gets the following billiard wall forms w_Γ ($\Gamma = 1, \cdots, 12$),

- Symmetry wall forms

$$\beta'^2 - \beta'^1 \,, \qquad \beta'^3 - \beta'^2 \,, \qquad \beta'^3 - \beta'^1 \,. \tag{4.1}$$

- Curvature wall forms

$$2\beta'^1 \,, \qquad 2\beta'^2 \,, \qquad 2\beta'^3 \,. \tag{4.2}$$

[3] In spacetime dimension D, the dominant walls include always the symmetry walls $\beta'^i - \beta'^j$ ($i > j$) where β'^i are the scale factors in D dimensions. If $D = d + 1 > 3$ is the endpoint of the oxidation sequence, $\beta'^d - \beta'^{d-1}$ becomes upon dimensional reduction the symmetry wall (3.10) (the indices $(1, 2)$ in 3 dimensions correspond to $(d-1, d)$ in D dimensions) [18] while $\beta'^{d-1} - \beta'^{d-2}$ becomes a wall of the form $\beta^1 - k(\phi)$ with $k(\phi)$ some linear form in the dilatons. Since this wall is dominant and since the only dominant wall in 3 dimensions of the form $\beta^1 - k(\phi)$ is the dominant magnetic wall (3.12), they must indeed be equal.

- Electric wall forms

$$\beta'^1\,, \qquad \beta'^2\,, \qquad \beta'^3\,. \tag{4.3}$$

- Magnetic wall forms

$$\beta'^1\,, \qquad \beta'^2\,, \qquad \beta'^3\,, \tag{4.4}$$

where the β'^i are the (logarithmic) scale factors in $D = 4$ spacetime dimensions.

The billiard is clearly defined by the subset of inequalities

$$\alpha_1(\beta') \equiv \beta'^1 \geq 0\,, \qquad \alpha_2(\beta') \equiv \beta'^2 - \beta'^1 \geq 0\,, \qquad \alpha_3(\beta') \equiv \beta'^3 - \beta'^2 \geq 0\,, \tag{4.5}$$

since the other inequalities are obvious consequences of this subset. A straightforward calculation, using the metric in the space of the scale factors,

$$\sum_{i=1}^{3} \left(d\beta'^i\right)^2 - \left(\sum_{i=1}^{3} d\beta'^i\right)^2 \tag{4.6}$$

or more properly, its inverse

$$\sum_{i=1}^{3} (\partial_i f)^2 - \frac{1}{2}\left(\sum_{i=1}^{3} \partial_i f\right)^2 \tag{4.7}$$

shows that the matrix

$$A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}$$

is equal to

$$A_{ij} = \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{4.8}$$

The 2×2 submatrix

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \tag{4.9}$$

is the Cartan matrix of $A_2^{(2)}$ [12], the second root α_2 being the long root. The third root α_3 is clearly attached to α_2 with a single line, which enables one to identify (4.8) with the Cartan matrix of the lorentzian extension $A_2^{(2)\wedge}$ of the twisted affine algebra $A_2^{(2)}$.

If instead of the Einstein-Maxwell theory, we had the pure Einstein theory, the electromagnetic electric and magnetic walls would be absent and the billiard would be defined by the subset of inequalities

$$\tilde{\alpha}_1(\beta') \equiv 2\beta'^1 \geq 0\,, \qquad \alpha_2(\beta') \equiv \beta'^2 - \beta'^1 \geq 0\,, \qquad \alpha_3(\beta') \equiv \beta'^3 - \beta'^2 \geq 0 \tag{4.10}$$

instead of (4.5), $\tilde{\alpha}_1$ being the dominant curvature wall form. Both (4.10) and (4.5) define the same billiard but the normalization of the first root, which is an information contained in the lagrangian [1], is different. The Cartan matrix associated with (4.10) is

$$\tilde{A}_{ij} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{4.11}$$

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This is the Cartan matrix of $A_1^{\wedge\wedge}$ [7]. The passage from $A_1^{\wedge\wedge}$ to $A_2^{(2)\wedge}$ when one includes the Maxwell field comes from the fact that the curvature root $\tilde{\alpha}_1$, which is dominant in the absence of electromagnetism, ceases to be so and is replaced by α_1 ($\tilde{\alpha}_1 = 2\alpha_1$). The two algebras have clearly the same fundamental Weyl chamber and Weyl group.

We also see directly on the 4-dimensional formulation that the short root α_1 is twice degenerate, it appears once as electric wall form and once as magnetic wall form (because of electric-magnetic duality, the electric and magnetic energy densities contribute the same wall forms). Furthermore, twice the root α_1 is also a root (in fact, a curvature root), another feature characteristic of the non reduced restricted root system of bc_1-type. This matches of course perfectly the computation in 3 spacetime dimensions. When reduced to $D = 3$ spacetime dimensions, the $D = 4$ Einstein-Maxwell system yields a $\mathrm{SU}(2,1)$ coset model $(\mathrm{SU}(2,1)/\,\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)))$ coupled to gravity [17]. The real rank is one and there is only one dilaton ϕ. The restricted root system of the coset is of bc_1-type [16]. By applying the standard formulas of dimensional reduction, one finds that the reduced action has indeed the form of (3.3), where the restricted roots are $\frac{1}{\sqrt{2}}\phi$ (degenerated twice) and $\sqrt{2}\phi$ (highest root with multiplicity one). The $3D$ electric wall is just the $4D$ electric (or magnetic) wall α_1 in (4.5); the $3D$ magnetic wall is the $4D$ symmetry wall α_2 in (4.5); and the $3D$ symmetry wall is the $4D$ symmetry wall α_3 in (4.5): everything matches once the $3D \to 4D$ translation is appropriately done.

5. Twisted case — link with real forms of affine algebras

As we have seen, adding the last overextended root is direct and involves no subtlety. The twist arises already — and only — at the level of the affinization. With this in mind, the appearance of twisted algebras in the description of the billiard is not surprizing if one recalls the theory of real forms of affine Kac-Moody algebras. For the "almost split" case their classification is given in [11]. By mere inspection of the tables given in that paper, it can be observed that if $\mathcal{F}$ is the maximal split subalgebra of the finite-dimensional real Lie algebra $\mathcal{G}$, the affine extension $\mathcal{F}^\wedge$ *may not be the maximal split subalgebra of* $\mathcal{G}^\wedge$. In particular, the maximal split subalgebra of the current algebras $su(2,1)^\wedge$ and $su(4,1)^\wedge$ is $A_2^{(2)}$, while the maximal split subalgebra of the current algebra $E_{6|-14}^\wedge$ is $A_4^{(2)}$. Note that for both $A_2^{(2)}$ and $A_4^{(2)}$, twice the shortest simple root is a root, a feature that the authors of [11] denote by adding a $\times$ over the short root on the restricted Dynkin diagram.

On the other hand, the maximal split subalgebras of the other current algebras relevant to the supergravity models under consideration, namely, $A_1^\wedge$, $so(8,k+2)^\wedge$ (with $0 \leq k < 6$), $so(8,8)^\wedge$, $so(8,k+2)^\wedge$ (with $k > 6$), $E_{7|-5}^\wedge$ and $E_{8|+8}^\wedge$ involve no twist and are respectively $A_1^\wedge$, $B_{k+2}^\wedge$, $D_8^\wedge$, $B_8^\wedge$, $F_4^\wedge$ and $E_8^\wedge \equiv E_{8|+8}^\wedge$.

6. Conclusions and summary

In this paper, we have established a theorem that enables one to derive the billiard dynamics of all $D = 4$ pure supergravity models directly from their $D = 3$ formulation: the relevant data that control the dynamics in the vicinity of a spacelike singularity are the

D=4 SUGRAS				
N	$\mathcal{U}_3$	$\bar{B}$	$\mathcal{F}$	$\mathcal{A}$
$N=1$	$sl(2,R)$	a_1	A_1	$A_1^{\wedge\wedge}$
$N=2$	$su(2,1)$	bc_1	A_1	$A_2^{(2)\wedge}$
$N=3$	$su(4,1)$	bc_1	A_1	$A_2^{(2)\wedge}$
$N=4$	$so(8,2)$	c_2	C_2	$C_2^{\wedge\wedge}$
$N=5$	$E_{6\vert-14}$	bc_2	C_2	$A_4^{(2)\wedge}$
$N=6$	$E_{7\vert-5}$	f_4	F_4	$F_4^{\wedge\wedge}$
$N=8$	$E_{8\vert+8}$	e_8	E_8	$E_8^{\wedge\wedge}$
$N=4,\ k<6$	$so(8,k+2)$	b_{k+2}	B_{k+2}	$B_{k+2}^{\wedge\wedge}$
$N=4,\ k=6$	$so(8,8)$	d_8	D_8	$D_8^{\wedge\wedge}$
$N=4,\ k>6$	$so(8,k+2)$	b_8	B_8	$B_8^{\wedge\wedge}$

Table 1: Billiard data for $D=4$ supergravities.

restricted root system and the maximal split subalgebra of the real $D=3$ symmetry algebra. The precise rule is: the Kac-Moody algebra $\mathcal{A}$ relevant for the billiard motion is just the overextension of the maximal split subalgebra $\mathcal{F}$, $\mathcal{A}=\mathcal{F}^{\wedge\wedge}$, except when the restricted root system is of bc-type, in which case there is a twist. Because the Kac-Moody algebras that emerge for the models are all hyperbolic, the dynamics is asymptotically chaotic.

Our theorem goes beyond $D=4$ pure supergravities and covers all systems whose reduction to $D=3$ is a non-linear sigma model G/H coupled to gravity. This is the case for $D=4$ supergravities with k vector ("Maxwell") multiplets, for which the $D=3$ symmetry algebra is $so(8,k+2)$. It is also the case, for instance, for the $N=2$ $D=5$ exceptional Einstein-Maxwell theories of [23, 24]. Since these latter models are described in $D=3$ by the real Lie algebras $F_{4|4}$, $E_{6|2}$, $E_{7|-5}$ or $E_{8|-24}$ (corresponding to the real, complex, quaternionic or octonionic Jordan algebras, respectively) [24], one can immediately infer that their billiard is the fundamental Weyl chamber of the hyperbolic Kac-Moody algebra $F_4^{\wedge\wedge}$. Indeed, F_4 is in all cases the maximal split subalgebra (and the restricted root system is of f_4 type).

Our results are summarized in the table 1, in which we give the number of supersymmetries (N), the real symmetry algebra that emerges in $D=3$ dimensions ($\mathcal{U}_3$), the corresponding restricted root system ($\bar{B}$), the maximal split subalgebra ($\mathcal{F}$) and the hyperbolic Kac-Moody algebra that controls the BKL limit ($\mathcal{A}$). In the last three lines, k is the number of coupled Maxwell multiplets (thus the fourth line corresponds to $k=0$). We denote the root systems by small letters to distinguish them from the Lie algebras. (Recall the equivalences $A_1\equiv B_1\equiv C_1$ and $B_2\equiv C_2$.) The real algebra $\mathcal{F}$ is by definition split, i.e., it always corresponds to the maximally non-compact real form, which is for that reason not explicitly written (so, in the $\mathcal{F}$-column, $A_1\equiv A_{1|+1}\equiv sl(2,R)$, $B_8\equiv so(8,9)$ etc). Similarly, the Kac-Moody algebra $\mathcal{A}$ in the last column is split (real linear combinations of the Chevalley generators and of their multiple commutators).

The restricted root system is of bc-type only for $N=2,3$ and $N=5$. There is then a twist. For $N=2$ and $N=3$, one gets $A_2^{(2)\wedge}$ instead of $A_1^{\wedge\wedge}$; while for $N=5$, it is

$A_4^{(2)\wedge}$ that appears rather than $C_2^{\wedge\wedge}$. Note that the actual twisted algebra that emerges has not only the same rank as the standard (untwisted) overextension $\mathcal{F}^{\wedge\wedge}$, but also the same Weyl group,

$$\begin{aligned} W(A_1^{\wedge\wedge}) &\simeq W(A_2^{(2)\wedge}) , \\ W(C_2^{\wedge\wedge}) &\simeq W(A_4^{(2)\wedge}) . \end{aligned} \tag{6.1}$$

One of the interests of the billiard analysis is its connection with U-dualities [25] and hidden symmetries of the theory, for which various proposals exist [17] and [26]–[30]. We reserve for further study a more detailed analysis of the significance of the twist in the symmetry structure of the models where it appears.

Finally, we note that although derived with the purpose of determining the billiard structure of supergravity theories, our theorem makes no use of supersymmetry. As stressed previously, the only relevant datum is the real symmetry group $\mathcal{U}_3$ which characterizes the manifold of the scalar fields coupled to gravity in the toroidal compactification of the theory to three dimensions.

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A. Overextensions of finite-dimensional simple Lie algebras

Let $\mathcal{L}$ be a complex, finite-dimensional, simple Lie algebra of rank r, with simple roots $\alpha_1, \alpha_2, \ldots, \alpha_r$. We normalize the roots so that the long roots have squared length equal to 2 (the short roots, if any, have then squared length equal to 1 (or 2/3 for G_2)). The roots of simply-laced algebras are regarded as long roots.

We denote by θ the highest root. It is a long root. We denote by V the r-dimensional euclidean vector space spanned by α_i ($i = 1, \ldots, r$). Let M_2 be the 2-dimensional Minkowski space with basis vectors u and v so that $(u|u) = (v|v) = 0$ and $(u|v) = 1$. The metric in the space $V \oplus M_2$ has clearly minkowskian signature $(-, +, +, \ldots, +)$ so that any Kac-Moody algebra whose simple roots span $V \oplus M_2$ is necessarily lorentzian.

A.1 Standard overextensions

The standard overextensions $\mathcal{L}^{\wedge\wedge}$ are obtained by adding to the original roots of $\mathcal{L}$ the roots

$$\alpha_0 = u - \theta \, , \qquad \alpha_{-1} = -u - v \, .$$

The root α_0 is called the affine root and the algebra $\mathcal{L}^\wedge(\equiv \mathcal{L}^{(1)})$ with roots $\alpha_0, \alpha_1, \cdots, \alpha_r$ is the untwisted affine extension of $\mathcal{L}$. The root α_{-1} is known as the overextended root. One has clearly $\text{rank}(\mathcal{L}^{\wedge\wedge}) = \text{rank}(\mathcal{L}) + 2$.

The algebras $A_k^{\wedge\wedge}$ ($k \leq 7$), $B_k^{\wedge\wedge}$ ($k \leq 8$), $C_k^{\wedge\wedge}$ ($k \leq 4$), $D_k^{\wedge\wedge}$ ($k \leq 8$), $G_2^{\wedge\wedge}$ $F_4^{\wedge\wedge}$, $E_k^{\wedge\wedge}$ ($k = 6, 7, 8$) are hyperbolic. (The list of hyperbolic KM algebras may be found in [31].)

A.2 Twisted overextensions

Twisted affine algebras are related to either the bc-root systems or to extensions by the highest short root (see [12, proposition 6.4]).

A.2.1 Twisted overextensions associated with the bc-root systems

These are the overextensions met in the text. The construction proceeds as for the untwisted overextensions, but the starting point is now the bc_r root system. The restricted Dynkin diagram of bc_r is the Dynkin diagram of B_r with a $\times$ over the simple short root, say α_1, to indicate that $2\alpha_1$ is also a root. The roots are also rescaled by the factor $(1/\sqrt{2})$ so that the highest root θ of the bc-system has length 2 (instead of 4). Indeed, θ is given by $\theta = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_r)$ [16]. It has squared length equal to 2 (with the rescaling) and has non-vanishing scalar product only with α_r ($(\alpha_r|\theta) = 1$). The overextension procedure yields the algebra $BC_r^{\wedge\wedge} \equiv A_{2r}^{(2)\wedge}$.

There is an alternative overextension $A_{2r}^{(2)'\wedge}$ that can be defined by starting this time with the algebra C_r but taking *one-half* the highest root of C_r to make the extension (see [12, formula in paragraph 6.4, bottom of page 84]). The formulas for α_0 and α_{-1} are $2\alpha_0 = u - \theta$ and $2\alpha_{-1} = -u - v$ (where θ is now the highest root of C_r). The Dynkin diagram of $A_{2r}^{(2)'\wedge}$ is (Langlands) dual to that of $A_{2r}^{(2)\wedge}$. (Duality amounts to reversing the arrows in the Dynkin diagram, i.e., to replacing the (generalized) Cartan matrix by its transpose.)

The algebras $A_{2r}^{(2)\wedge}$ and $A_{2r}^{(2)'\wedge}$ have rank $r + 2$ and are hyperbolic for $r \leq 4$. The intermediate affine algebras are in all cases the twisted affine algebras $A_{2r}^{(2)}$. By coupling to 3-dimensional gravity a coset model G/H where the restricted root system of the (real) Lie algebra of the Lie group G is of bc_r-type, one can realize all the $A_{2r}^{(2)\wedge}$ algebras.

A.2.2 Twisted overextensions associated with the highest short root

We denote by θ_s the unique short root of greatest height. It exists only for non-simply laced algebras and has length 1 (or 2/3 for G_2). The twisted overextensions are defined as the standard overextensions but one uses instead the highest short root θ_s. The formulas for the affine and overextended roots are

$$\alpha_0 = u - \theta_s \,, \qquad \alpha_{-1} = -u - \frac{1}{2}v \,, \qquad (\mathcal{L} = B_r, C_r, F_4)$$

or

$$\alpha_0 = u - \theta_s \,, \qquad \alpha_{-1} = -u - \frac{1}{3}v \,, \qquad (\mathcal{L} = G_2) \,.$$

(We choose the overextended root to have the same length as the affine root and to be attached to it with a single link. This choice is motivated by considerations of simplicity and yields the fourth rank ten hyperbolic algebra when $\mathcal{L} = C_8$.)

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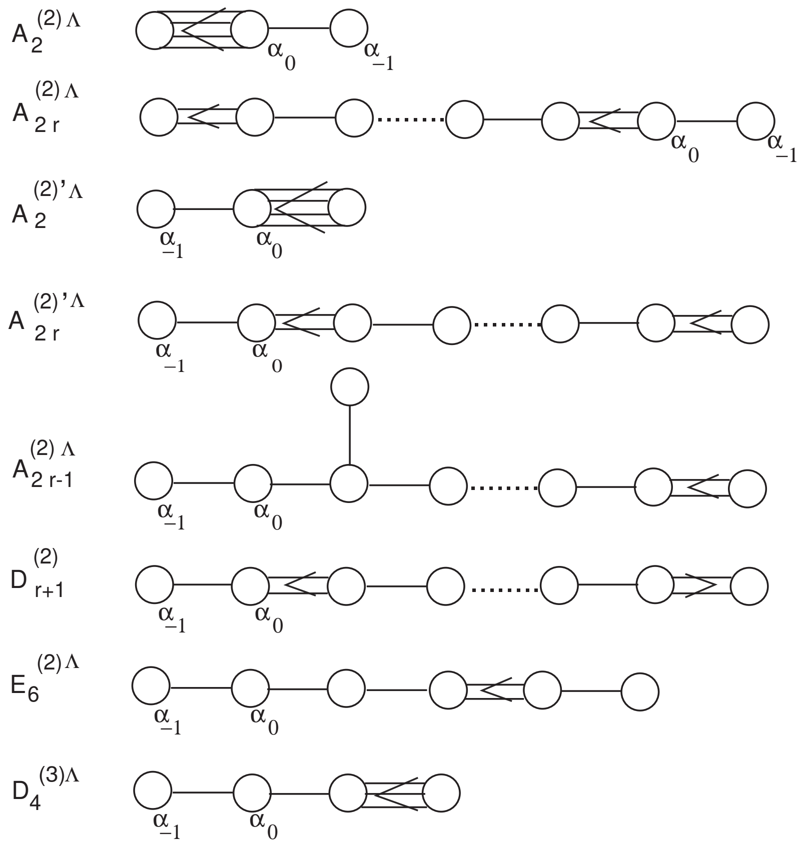

Figure 2: Twisted overextensions of finite-dimensional, simple Lie algebras.

The affine extensions generated by $\alpha_0, \cdots, \alpha_r$ are respectively the twisted affine algebras $D^{(2)}_{r+1}$ ($\mathcal{L} = B_r$), $A^{(2)}_{2r-1}$ ($\mathcal{L} = C_r$), $E^{(2)}_6$ ($\mathcal{L} = F_4$) and $D^{(3)}_4$ ($\mathcal{L} = G_2$). The overextensions $D^{(2)\wedge}_{r+1}$ have rank $r+2$ and are hyperbolic for $r \leq 4$. The overextensions $A^{(2)\wedge}_{2r-1}$ have rank $r+2$ and are hyperbolic for $r \leq 8$. The last hyperbolic case, $r = 8$, yields the algebra $A^{(2)\wedge}_{15}$ also denoted CE_{10}. It is the fourth rank-10 hyperbolic algebra, besides E_{10}, BE_{10} and DE_{10}. (CE_{10} is also considered in [10].) The overextensions $E^{(2)\wedge}_6$ (rank 6) and $D^{(3)\wedge}_4$ (rank 4) are hyperbolic.

A.2.3 Dynkin diagrams

We list below the Dynkin diagrams of all twisted overextensions. A satisfactory feature of the class of overextensions (standard *and* twisted) is that it is closed under duality. For instance, $A^{(2)\wedge}_{2r-1}$ is dual to $B^{\wedge\wedge}_r$. In fact, one could get the twisted overextensions associated with the highest short root from the standard overextensions precisely by requiring closure under duality. A similar feature already holds for the affine algebras.

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